Linear Algebra and its Applications 433 (2010) 380–389

Contents lists available at ScienceDirect

Linear Algebra and its Applications

journal homepage: www.elsevier.com/locate/laa

# On the weighted trees with given degree sequence and positive weight set ☆

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ARTICLE INFO

*Article history:*
Received 23 November 2009
Accepted 4 March 2010
Available online 27 March 2010

Submitted by R.A. Brualdi

*AMS classification:*
05C50

*Keywords:*
Weighted tree
Spectral radius
Degree sequence
Perron vector

ABSTRACT

We determine the (unique) weighted tree with the largest spectral radius with respect to the adjacency and Laplacian matrix in the set of all weighted trees with a given degree sequence and positive weight set. Moreover, we also derive the weighted trees with the largest spectral radius with respect to the matrices mentioned above in the sets of all weighted trees with a given maximum degree or pendant vertex number and so on.



## 1. Introduction

In the paper we only consider simple weighted graphs with a positive weight set. Let $G$ be a weighted graph with vertex set $\{v_1, v_2, \ldots, v_n\}$, edge set $E(G) \neq \emptyset$ and weight set $W(G) = \{w_j > 0 : j = 1, 2, \ldots, |E(G)|\}$. The function $w_G : E(G) \to W(G)$ is called a *weight function* of $G$. It is obvious that each weighted graph corresponds to a weight function. For convenience, define $w_G(uv) = 0$ if $uv \notin E(G)$. Then $G$ may be regarded as a weighted graph with a nonnegative weight set, where $uv \in E(G)$ if and only if $w_G(uv) > 0$. So the *adjacency matrix* of $G$ is the $n \times n$ matrix $A(G) = (w_G(v_iv_j))$.

The weight of $v_i$, denoted by $w_G(v_i)$, is the sum of weights of all edges incident to $v_i$ in $G$. Let $W(G) = \mathrm{diag}(w_G(v_1), w_G(v_2), \ldots, w_G(v_n))$ be the diagonal matrix of vertex weights of $G$. Then the *Laplacian matrix* of $G$ is $L(G) = W(G) - A(G)$.

☆ Supported by the National Natural Science Foundation of China (10871204).

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doi:10.1016/j.laa.2010.03.009

Since both $A(G)$ and $L(G)$ are real symmetric matrices, their eigenvalues are all real numbers. It is easy to see that their largest eigenvalues are positive numbers. We call their largest eigenvalues the *adjacent and Laplacian spectral radius* of $G$, denoted by $\rho(G)$ and $\mu(G)$, respectively.

Two weighted graphs $G$ and $H$ are called *isomorphic*, denoted by $G = H$, if there is a bijection $f$ from $V(G)$ to $V(H)$ such that $ab \in E(G)$ if and only if $f(a)f(b) \in E(H)$, and $w_G(ab) = w_H(f(a)f(b))$ for each $ab \in E(G)$. The *degree* of vertex $v$ of $G$, denoted by $d_G(v)$, is the number of edges incident to $v$ in $G$. $v$ is called a *pendant vertex* of $G$ if $d_G(v) = 1$. The *degree sequence* $(d_1, d_2, \ldots, d_n)$ of $G$ is the sequence of (vertex) degrees ordered, say, in a non-increasing way. All other concepts and notations not given in the paper are standard terminology of graph theory (see, for example, [1]).

An *unweighted graph* is a weighted graph with each of the edges bearing weight 1. Many recent results provide upper and lower bounds for the adjacent and Laplacian spectral radius of unweighted graphs with given some information of graphs such as the matching number, independence number, maximum degree, pendant vertices number or entire degree sequence, etc. (see, for example, [2–9]).

Let $\Theta(d_1, d_2, \ldots, d_n)$ be the set of unweighted connected graphs with a given degree sequence $(d_1, d_2, \ldots, d_n)$. The Brualdi–Solheid problem (BSP for short) put forward the determination of graphs maximizing the adjacent spectral radius in a given set of graphs. For $\Theta(d_1, d_2, \ldots, d_n)$, the BSP has not been solved in general but has some results for some special graphs such as trees and unicyclic graphs [8,10]. Moreover, for the special cases of caterpillars [11] and cycles with spikes [12] the similar results to those of [8,10] also have been obtained.

There is now an extensive literature that investigates the spectrum of adjacent or Laplacian matrix of weighted graphs. In particular, much attention is focused on investigating the upper bounds of adjacent or Laplacian spectral radius of weighted graphs (see, for example, [13–17]). For some special weighted graphs, say, weighted trees, there are also some literatures that investigate the similar problem to BSP, i.e., they have determined the weighted trees with the largest adjacent or Laplacian spectral radius in the set of all weighted trees with a given positive weight set and some information of graphs such as diameter, pendant vertex number and so on [18–20].

The remainder of this paper is organized as follows. Let $\Gamma(D_n, W_n)$ denote the set of all weighted trees with a given degree sequence $D_n = (d_1, d_2, \ldots, d_n)$ such that $d_1 \geqslant d_2 \geqslant \cdots \geqslant d_n$ and a given positive weight set $W_n = \{w_1, w_2, \ldots, w_{n-1}\}$. In Section 2, we determine the (unique) weighted tree with the largest adjacent spectral radius in $\Gamma(D_n, W_n)$. In Section 3, we determine the (unique) weighted tree with the largest Laplacian spectral radius in $\Gamma(D_n, W_n)$. Moreover, we also determine the weighted trees with the largest adjacent and Laplacian spectral radius in the sets of all weighted trees with a fixed positive weight set and a fixed maximum degree or a fixed pendant vertex number and so on.

## 2. On the weighted trees with the largest adjacent spectral radius

For any weighted graph $G$, since $A(G)$ is nonnegative, there is a nonnegative eigenvector corresponding to $\rho(G)$. In particular, when $G$ is connected, $A(G)$ is irreducible and by the well-known Perron–Frobenius Theorem (see, for example, [21]), $\rho(G)$ is simple and there is a unique positive unit eigenvector. We shall refer to such an eigenvector $x$ as the *adjacent Perron vector* of $G$ and denote the component corresponding to the vertex $v$ of $G$ by $x_v$ ($x_v$ is called *$\rho$-weight* of $v$ with respect to $x$ in [10]).

The following are three results about the perturbations of adjacent spectral radius of weighted graphs [18]. The second one, Lemma 2.2, is a generalization about the perturbation known as the *rotation* or *shifting* of unweighted graphs, the third one, Lemma 2.3, is a generalization about the perturbation known as the *switching* of unweighted graphs (see, for example, [8,12]).

**Lemma 2.1.** *Let $uv$ and $ab$ be distinct edges of a connected weighted graph $G$ and let $x$ be the adjacent Perron vector of $G$. For $0 < \delta \leqslant w_G(uv)$, let $G^1$ be the weighted graph obtained from $G$ such that*

$$w_{G^1}(uv) = w_G(uv) - \delta, \quad w_{G^1}(ab) = w_G(ab) + \delta,$$
$$w_{G^1}(e) = w_G(e), \quad e \in E(G) - \{ab, uv\}.$$

*If $x_u x_v \leqslant x_a x_b$, then $\rho(G) < \rho(G^1)$.*

**Lemma 2.2.** *Let $u$ and $v$ be two distinct vertices of a connected weighted graph $G$. Let $u_1, u_2, \ldots, u_s (u_i \neq v, s \neq 0)$ be some vertices of $N_G(u) \backslash N_G(v)$ and let $x$ be the adjacent Perron vector of $G$. Let $G^2$ be the weighted graph obtained from $G$ by deleting the edges $uu_j$ and adding the edges $vu_j$ such that*

$$w_{G^2}(vu_j) = w_G(uu_j), \quad w_{G^2}(e) = w_G(e), \quad e \neq uu_j, \quad j = 1, 2, \ldots, s.$$

*If $x_v \geqslant x_u$, then $\rho(G) < \rho(G^2)$.*

**Lemma 2.3.** *Let $uv$, $ab$, $ub$ and $av$ be distinct edges of a weighted connected graph $G$ and let $x$ be the adjacent Perron vector of $G$. For $0 < \delta \leqslant w_G(uv)$ and $0 < \theta \leqslant w_G(ab)$, let $G^3$ be the weighted graph obtained from $G$ such that*

$$w_{G^3}(uv) = w_G(uv) - \delta, \quad w_{G^3}(ub) = w_G(ub) + \delta, \quad w_{G^3}(ab) = w_G(ab) - \theta,$$
$$w_{G^3}(av) = w_G(av) + \theta, \quad w_{G^3}(e) = w_G(e), \quad e \in E(G) - \{uv, ab, ub, av\}.$$

*If $(x_b - x_v)(\delta x_u - \theta x_a) \geqslant 0$, then $\rho(G) \leqslant \rho(G^3)$, and with the equality $\rho(G) = \rho(G^3)$ if and only if $x_b = x_v$ and $\delta x_u = \theta x_a$.*

Assume that $T_M$ is a weighted tree in $\Gamma(D_n, W_n)$ with the largest adjacent spectral radius and always suppose that $x$ is the adjacent Perron vector of $T_M$. We also assume that $n \geqslant 4$ (otherwise, $T_M$ is a weighted star, so the problem is trivial).

The following result is a corollary of Lemma 2.1, which will be used to compare the weights of edges of $T_M$ by means of the adjacent Perron vector of $T_M$.

**Lemma 2.4.** *Let $ab$ and $uv$ be two distinct edges of $T_M$.*

(1) *If $x_a x_b \geqslant x_u x_v$, then $w_{T_M}(ab) \geqslant w_{T_M}(uv)$.*
(2) *If $w_{T_M}(ab) > w_{T_M}(uv)$, then $x_a x_b > x_u x_v$.*
(3) *If $x_a x_b = x_u x_v$, then $w_{T_M}(ab) = w_{T_M}(uv)$.*

**Proof.** It is easy to see that (2) and (3) can be immediately deduced from (1). Hence we only prove (1). Assume that $w_{T_M}(ab) < w_{T_M}(uv)$. Put $\delta = w_{T_M}(uv) - w_{T_M}(ab)$ and let $T'$ be the weighted tree obtained from $T_M$ such that

$$w_{T'}(ab) = w_{T_M}(ab) + \delta, \quad w_{T'}(uv) = w_{T_M}(uv) - \delta, \quad w_{T'}(e) = w_{T_M}(e), \quad e \neq ab, uv,$$

namely $T'$ is the weighted tree obtained from $T_M$ by exchanging the weights of edges $ab$ and $uv$ while making the weights of other edges unchanged. Then $T' \in \Gamma(D_n, W_n)$, and by Lemma 2.1, $\rho(T') > \rho(T_M)$, a contradiction with the choice of $T_M$. $\square$

**Lemma 2.5.** *Let $P$ be a path of $T_M$ and let $a$, $b$, $u$, $v$ be four serial distinct vertices of $P$ such that $ab$, $uv \in E(T_M)$. Then*

(1) *$(x_u - x_b)(w_{T_M}(ab)x_a - w_{T_M}(uv)x_v) \leqslant 0$. In addition, $x_u = x_b$ if and only if $w_{T_M}(ab)x_a = w_{T_M}(uv)x_v$.*
(2) *$(x_v - x_a)(w_{T_M}(ab)x_b - w_{T_M}(uv)x_u) \leqslant 0$. In addition, $x_v = x_a$ if and only if $w_{T_M}(ab)x_b = w_{T_M}(uv)x_u$.*

**Proof.** We only prove (1). Assume the contrary, that is

$$(x_u - x_b)(w_{T_M}(ab)x_a - w_{T_M}(uv)x_v) > 0,$$

or that only one between $x_u = x_b$ and $w_{T_M}(ab)x_a = w_{T_M}(uv)x_v$ holds. Take $\theta = w_{T_M}(ab)$, $\delta = w_{T_M}(uv)$. Let $T'$ be the weighted tree obtained from $T_M$ such that

$$w_{T'}(ab) = w_{T_M}(ab) - \theta, \quad w_{T'}(au) = w_{T_M}(au) + \theta, \quad w_{T'}(uv) = w_{T_M}(uv) - \delta,$$
$$w_{T'}(vb) = w_{T_M}(vb) + \delta, \quad w_{T'}(e) = w_{T_M}(e), \quad e \in E(T_M) \backslash \{ab, uv\},$$

i.e., $T'$ is the weighted tree obtained from $T_M$ by deleting the edges $ab$, $uv$ and adding the edges $au$, $vb$ such that $w_{T'}(au) = w_{T_M}(ab)$, $w_{T'}(vb) = w_{T_M}(uv)$. Then $T' \in \Gamma(D_n, W_n)$, and from Lemma 2.3, $\rho(T') > \rho(T_M)$, a contradiction with the choice of $T_M$. □

**Lemma 2.6.** *Let $P = z_1z_2\cdots z_s$ be a path of $T_M$ such that $d_{T_M}(z_1) = d_{T_M}(z_s) = 1$ and $z_iz_{i+1} \in E(T_M)$, $i = 1, 2, \ldots, s-1$. Assume that there are two vertices $z_r$ and $z_l(r < l)$ with $x_{z_r} = x_{z_l}$. Then $x_{z_{r+1}} = x_{z_{l-1}}$ for $l - r \geqslant 3$ and $x_{z_{r-1}} = x_{z_{l+1}}$ for $r > 1$ and $l < s$.*

**Proof.** First assume $x_{z_{r+1}} \neq x_{z_{l-1}}$ for $l - r \geqslant 3$. Without loss of generality, assume $x_{z_{r+1}} > x_{z_{l-1}}$. Write $a = z_r$, $b = z_{r+1}$, $u = z_{l-1}$, $v = z_l$. Then $x_ax_b > x_ux_v$. By Lemma 2.4(1), we have that $w_{T_M}(ab) \geqslant w_{T_M}(uv)$. If $w_{T_M}(ab) > w_{T_M}(uv)$, then

$$x_a = x_v, \quad w_{T_M}(ab)x_b > w_{T_M}(uv)x_u. \tag{2.1}$$

If $w_{T_M}(ab) = w_{T_M}(uv)$, then

$$x_b > x_u, \quad w_{T_M}(ab)x_a = w_{T_M}(uv)x_v. \tag{2.2}$$

Both (2.1) and (2.2) contradict with the additional claims of Lemma 2.5. Therefore, we have $x_{z_{r+1}} = x_{z_{l-1}}$.

Next assume $x_{z_{r-1}} \neq x_{z_{l+1}}$ for $r > 1$ and $l < s$. Without loss of generality, assume $x_{z_{r-1}} > x_{z_{l+1}}$. Let $a = z_{r-1}$, $b = z_r$, $u = z_l$, $v = z_{l+1}$. Then $x_ax_b > x_ux_v$. By Lemma 2.4(1), we have that $w_{T_M}(ab) \geqslant w_{T_M}(uv)$. If $w_{T_M}(ab) > w_{T_M}(uv)$, then

$$x_b = x_u, \quad w_{T_M}(ab)x_a > w_{T_M}(uv)x_v. \tag{2.3}$$

If $w_{T_M}(ab) = w_{T_M}(uv)$, then

$$x_a > x_v, \quad w_{T_M}(ab)x_b = w_{T_M}(uv)x_u. \tag{2.4}$$

Both (2.3) and (2.4) contradict with the additional claims of Lemma 2.5. Therefore, we have $x_{z_{r-1}} = x_{z_{l+1}}$. □

In order to characterize $T_M$ we need some concepts defined in [8]. To make the paper more self-contained, we will introduce them for a tree $T$. By means of breadth-first search a well-ordering of the vertices of $T$ can be determined as follows: choose a vertex $v_1$ of $T$ and create a sorted list of vertices beginning with $v_1$; append all neighbors $v_2, v_3, \ldots, v_{d_T(v_1)+1}$ of $v_1$ sorted by decreasing degrees; then append all neighbors of $v_2$ that are not already in this list; continue recursively with $v_3, v_4, \ldots$ until all vertices of $T$ are processed. In this way we build a well-ordering of the vertices of $T$ and some layers, where each vertex $v$ in layer $i$ has distance $i - 1$ from root $v_1$ which we call its *height*, denoted by $h(v)$. Moreover, $v$ in layer $i$ is adjacent to the unique vertex $w$ in layer $i - 1$. We call $w$ the *parent* of $v$ and $v$ a *child* of $w$.

Let $T$ be a weighted tree with vertex set $V$, edge set $E$ and positive weight set $W$. Then a well-ordering $\prec$ of the vertices of $T$ is called *the weighted breadth-first search ordering with decreasing*

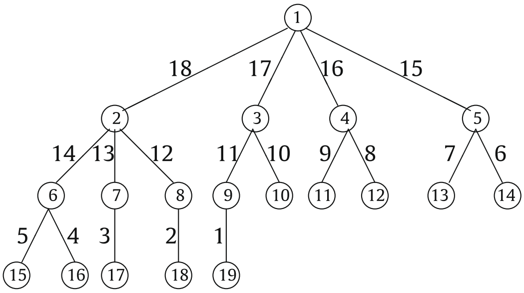


**Fig. 1.** An example of the WBFD-tree in $\Gamma(D_{19}, W_{19})$.

*degrees and weights* (*WBFD-ordering* for short) if the following holds for all vertices $v,\ u \in V$ and all edges $uv,\ ab \in E$:

(C1) if $u_1 \prec u_2$ then $v_1 \prec v_2$ for all children $v_1$ of $u_1$ and $v_2$ of $u_2$, respectively;
(C2) if $v \prec u$ then $d_T(v) \geqslant d_T(u)$;
(C3) if $u \prec v,\ a \prec b$ and $u \prec a$ then $w_T(uv) \geqslant w_T(ab)$.

A weighted tree is called a *WBFD-tree* if its vertices has a WBFD-ordering. For a given degree sequence and positive weight set, from (C1) and (C2) it is easy to see that the WBFD-tree is uniquely determined up to isomorphism.

**Example.** Let $a^{(k)} = \overbrace{a, a, \ldots, a}^{k}$. In Fig. 1, the WBFD-tree in $\Gamma(D_{19}, W_{19})$ is displayed, where the number on each edge denotes the weight corresponding to the edge, the number in each cycle denotes the position corresponding to the vertex in the WBFD-ordering $\prec$, while

$$D_{19} = (4^{(2)}, 3^{(4)}, 2^{(3)}, 1^{(10)}), \quad W_{19} = \{1, 2, 3, \ldots, 18\}.$$

The following is one of main results in this paper.

**Theorem 2.7.** *The WBFD-tree is the unique graph in $\Gamma(D_n, W_n)$ having the largest adjacent spectral radius.*

**Proof.** Create an ordering $\prec$ of vertices in $T_M$ by breadth-first search as follows: choose a vertex $v_1$ of $T_M$ with the largest component in $x$ as root layer 1; append all neighbors $v_2, v_3, \ldots, v_{d_{T_M}(v_1)+1}$ of $v_1$ to the list ordered list; these neighbors are ordered such that $u \prec v$ if $x_u \geqslant x_v$ (note that the ordering between $u$ and $v$ may be arbitrary if $x_u = x_v$); then continue recursively for all $v_2, v_3, \ldots$ until all vertices of $G$ are processed. So we get a rooted tree $T_M$ with the root $v_1$. The following Claim 1 is obvious.

**Claim 1.** *The ordering $\prec$ satisfies (C1).*

**Claim 2.** *If $\alpha \prec \beta$ then $x_\alpha \geqslant x_\beta$.*

We prove this result step by step by the following three claims.

**Claim 2.1.** *If $h(\alpha) = h(\beta)$ then $x_\alpha \geqslant x_\beta$.*

Assume $x_\alpha < x_\beta$. Let $P$ be a longest path of $T_M$ containing $\alpha$ and $\beta$. Write

$$P = z_1 z_2 \cdots z_r \cdots z_l \cdots z_s,$$

where $z_r = \alpha,\ z_l = \beta,\ z_1$ and $z_s$ are two pendant vertices of $T_M$. Since the vertices of the same layer in $T_M$ are not adjacent, by $h(z_r) = h(z_l)$, it is easy to see that the number $l - r + 1$ of vertices on $P$ between $z_r$ and $z_l$, including $z_r$ and $z_l$, is odd. If $z_{r+1} \neq z_{l-1}$, then by the assumption $x_{z_r} < x_{z_l}$ and Lemma 2.6 we have $x_{z_{r+1}} \neq x_{z_{l-1}}$. If $x_{z_{r+1}} > x_{z_{l-1}}$, then set $v = z_r,\ u = z_{r+1},\ b = z_{l-1},\ a = z_l$. So by Lemma 2.5, we get

$$(x_u - x_b)\left(w_{T_M}(ab)x_a - w_{T_M}(uv)x_v\right) \leqslant 0, \tag{2.5}$$
$$(x_v - x_a)\left(w_{T_M}(ab)x_b - w_{T_M}(uv)x_u\right) \leqslant 0. \tag{2.6}$$

Note that $x_u > x_b$ and $x_a > x_v$. So from (2.5) and (2.6), we get, respectively, that

$$w_{T_M}(ab) < \frac{x_v}{x_a} \cdot w_{T_M}(uv) < w_{T_M}(uv) \tag{2.7}$$

and

$$w_{T_M}(ab) > \frac{x_u}{x_b} \cdot w_{T_M}(uv) > w_{T_M}(uv), \tag{2.8}$$

a contradiction. Thus $x_{z_{r+1}} < x_{z_{l-1}}$. If $z_{r+2} \neq z_{l-2}$, then replacing the vertices $z_r$ and $z_l$ above by $z_{r+1}$ and $z_{l-1}$, respectively, in the similar way above we can get $x_{z_{r+2}} < x_{z_{l-2}}$. By proceeding in this way, we can obtain a $t$ such that $z_{r+t} \neq z_{l-t}$, $x_{z_{r+t}} < x_{z_{l-t}}$ and $z_{r+t+1} = z_{l-t-1}$. Since the ordering $\prec$ satisfies (C1) from Claim 1, by $z_r = \alpha \prec \beta = z_l$, we have $z_{r+i} \prec z_{l-i}$ $(i = 1, 2, \ldots, t)$. In particular, $z_{r+t} \prec z_{l-t}$. Note that $z_{r+t}$ and $z_{l-t}$ are two children of $z_{r+t+1} = z_{l-t-1}$, so by the the definition of the ordering $\prec$, we have $x_{z_{r+t}} \geqslant x_{z_{l-t}}$, a contradiction with $x_{z_{r+t}} < x_{z_{l-t}}$.

**Claim 2.2.** *If $h(\beta) = h(\alpha) + 1$ then $x_\alpha \geqslant x_\beta$.*

Assume the contrary, namely $x_\alpha < x_\beta$. We distinguish the following two cases.

**Case 1.** Assume that $\beta$ is a child of $\alpha$.

Let $P$ be a longest path of $T_M$ containing $\alpha$, $\beta$ and $v_1$. Write

$$P = z_1 z_2 \cdots z_r z_{r+1} \cdots z_l \cdots z_s,$$

where $z_r = \beta$, $z_{r+1} = \alpha$, $z_l = v_1$, $z_1$ and $z_s$ are two pendant vertices of $T_M$. We again distinguish the following two subcases.

**Case 1.1.** Assume that $\beta$ is a pendant vertex of $T_M$.

Write $N_{T_M}(\alpha) = \{z_r, z_{r+2}, a_1, \ldots, a_s\}$. Let $T'$ be the weighted tree obtained from $T_M$ by deleting the edges $\alpha z_{r+2}$, $\alpha a_i$ and adding the new edges $\beta z_{r+2}$, $\beta a_i$ such that

$$w_{T'}(\beta z_{r+2}) = w_{T_M}(\alpha z_{r+2}), \quad w_{T'}(\beta a_i) = w_{T_M}(\alpha a_i), \quad i = 1, 2, \ldots, s,$$
$$w_{T'}(e) = w_{T_M}(e), \quad e \in E(T_M) - \{\alpha z_{r+2}, \alpha a_i : i = 1, 2, \ldots, s\}.$$

Then $T' \in \Gamma(D_n, W_n)$, and from Lemma 2.2 we have $\rho(T') > \rho(T_M)$, a contradiction with the choice of $T_M$.

**Case 1.2.** Assume that $\beta$ is not a pendant vertex of $T_M$.

From the assumption we have $x_{z_r} = x_\beta > x_\alpha = x_{z_{r+1}}$. Again by Lemma 2.6 we have $x_{z_{r-1}} \neq x_{z_{r+2}}$. If $x_{z_{r+2}} > x_{z_{r-1}}$, then set $a = z_{r+2}$, $b = \alpha$, $u = \beta$, $v = z_{r-1}$. So we still have (2.5) and (2.6). Note that $x_u > x_b$ and $x_a > x_v$ still hold. So from (2.5) and (2.6), we also get (2.7) and (2.8), a contradiction. Therefore, $x_{z_{r+2}} < x_{z_{r-1}}$. If $z_{r+2} \neq v_1$ and $z_{r-1}$ is not a pendant vertex of $T_M$, then replacing the vertices $z_r$ and $z_{r+1}$ above by $z_{r-1}$ and $z_{r+2}$, respectively, in the similar way above we can get $x_{z_{r+3}} < x_{z_{r-2}}$. By proceeding in this way, we can get a $t$ such that either $z_{r-t}$ is a pendant vertex of $T_M$ or $z_{r+t+1} = v_1$ and $x_{z_{r+t+1}} < x_{z_{r-t}}$. If $z_{r-t}$ is a pendant vertex of $T_M$ then a contradiction will be yielded by the similar proof to Case 1.1. If $z_{r+t+1} = v_1$ then this contradicts with the choice $v_1$.

**Case 2.** Assume that $\beta$ is not a child of $\alpha$.

Let $\gamma$ be the parent of $\beta$. We again distinguish the following two subcases.

**Case 2.1.** Assume that $\alpha \prec \gamma$.

Since $\alpha \prec \gamma$ and $h(\alpha) = h(\gamma)$, by Claim 2.1 we have $x_\alpha \geqslant x_\gamma$. Again from the assumption $x_\beta > x_\alpha$, we get $x_\beta > x_\gamma$. This will yield a contradiction by the similar proof of Case 1.

**Case 2.2.** Assume that $\gamma \prec \alpha$.

From $x_\alpha < x_\beta$ and the choice of $v_1$, it follows that $\alpha \neq v_1$. Let $P$ be a longest path of $T_M$ containing $\alpha$ and $\beta$. Write $P = z_1 z_2 \cdots z_r \cdots z_l \cdots z_s$, where $z_r = \alpha$, $z_l = \beta$, $z_1$ and $z_s$ are two pendant vertices of $T_M$. From the assumption we have $x_{z_r} = x_\alpha < x_\beta = x_{z_l}$. So from Lemma 2.6 we have $x_{z_{r+1}} \neq x_{z_{l-1}}$. Assume $x_{z_{r+1}} > x_{z_{l-1}}$. If $z_{l-2}$ is not the parent of $z_r$, then set $a = z_{r+1}$, $b = \alpha$, $u = \beta$, $v = z_{l-1}$. So we still have (2.5) and (2.6). Since $x_u > x_b$ and $x_a > x_v$ still hold, we also get (2.7) and (2.8), a contradiction. Hence $x_{z_{r+1}} < x_{z_{l-1}}$. If $z_{l-3}$ is not the parent of $z_{r+1}$, then replacing the vertices $z_l$ and $z_r$ above by $z_{l-1}$ and $z_{r+1}$, respectively, in the similar way above we can obtain $x_{z_{r+2}} < x_{z_{l-2}}$. By proceeding in this way, we can get a $t$ such that $x_{z_{r+t+1}} < x_{z_{l-t-1}}$ and $z_{l-t-2} = z_{r+t+1}$ is the parent of $z_{l-t-1}$ and $z_{r+t}$. Since $z_{r+t+1}$ is the parent of $z_{l-t-1}$, by the similar proof to Case 1 we have $x_{z_{r+t+1}} \geqslant x_{z_{l-t-1}}$, a contradiction.

**Claim 2.3.** *If $h(\beta) \geqslant h(\alpha) + 2$ then $x_\alpha \geqslant x_\beta$.*

Let $v_i$ be any vertex in layer $i$, $i = h(\alpha) + 1, h(\alpha) + 2, \ldots, h(\beta) - 1$. Then

$$\alpha \prec v_{h(\alpha)+1} \prec v_{h(\alpha)+2} \prec \cdots \prec v_{h(\beta)-1} \prec \beta.$$

So by Claim 2.2 we have $x_\alpha \geqslant x_{v_{h(\alpha)+1}} \geqslant x_{v_{h(\alpha)+2}} \geqslant \cdots \geqslant x_{v_{h(\beta)-1}} \geqslant x_\beta$.

**Claim 3.** *The ordering $\prec$ satisfies (C2).*

Let $v$ and $u$ be two arbitrary vertices of $T_M$ such that $v \prec u$. Then $x_v \geqslant x_u$ by Claim 2. Suppose that $d_{T_M}(v) < d_{T_M}(u)$. Write $s = d_{T_M}(u) - d_{T_M}(v)$. Let $u_1, u_2, \ldots, u_s$ be the neighbors of $u$ such that they are further from $v$ than $u$. Let $T'$ be the weighted tree obtained from $T_M$ by deleting the edges $uu_i$ and adding the edges $vu_i$ such that

$$w_{T'}(vu_i) = w_{T_M}(uu_i), \quad i = 1, 2, \ldots, s,$$
$$w_{T'}(e) = w_{T_M}(e), \quad e \in E(T_M) - \{uu_i : i = 1, 2, \ldots, s\}.$$

Then $T' \in \Gamma(D_n, W_n)$, and by Lemma 2.2 we have $\rho(T') > \rho(T_M)$, a contradiction with the choice of $T_M$. Therefore, $d_{T_M}(v) \geqslant d_{T_M}(u)$, namely the ordering $\prec$ satisfies (C2).

**Claim 4.** *The ordering $\prec$ satisfies (C3).*

Let $uv$ and $ab$ be two arbitrary edges of $T_M$ with $u \prec v$, $a \prec b$ and $u \prec a$. By Claim 1 we have $v \prec b$. By Claim 2 we have $x_u \geqslant x_a$ and $x_v \geqslant x_b$. So $x_u x_v \geqslant x_a x_b$. It follows that $w_{T_M}(uv) \geqslant w_{T_M}(ab)$ from Lemma 2.4(1), i.e., the ordering $\prec$ satisfies (C3).

By Claims 1, 3 and 4, $\prec$ is a WBFD-ordering, namely $T_M$ is a WBFD-tree. □

Let $\Omega(\Delta, W_n)$ be the set of all weighted trees with a given positive weight set $W_n$ and a maximum degree $\Delta$.

**Theorem 2.8**

(1) *If $\Delta = 2$ then the WBFD-tree in $\Gamma((2^{(n-2)}, 1^{(2)}), W_n)$ is the unique graph in $\Omega(\Delta, W_n)$ having the largest adjacent spectral radius.*

(2) *If $\Delta \geqslant 3$ then the WBFD-tree in $\Gamma((\Delta^{(s)}, t^{(p)}, 1^{(q)}), W_n)$ is the unique graph in $\Omega(\Delta, W_n)$ having the largest adjacent spectral radius, where $1 < t < \Delta$, $0 \leqslant p \leqslant 1$ and $s, p, q, t$ satisfy $s + p + q = n$, $s\Delta + pt + q = 2(n - 1)$.*

**Proof.** Let $T_M$ be a weighted tree in $\Omega(\Delta, W_n)$ with the largest adjacent spectral radius.

(1) If $\Delta = 2$ then the degree sequence of $T_M$ is $D_n = (2^{(n-2)}, 1^{(2)})$. It follows that $T_M \in \Gamma(D_n, W_n)$. So by Theorem 2.7 the WBFD-tree in $\Gamma(D_n, W_n)$ is the unique graph in $\Omega(\Delta, W_n)$ having the largest adjacent spectral radius.

(2) Let $\Delta \geqslant 3$. Suppose that there exist two vertices $v$ and $u$ of $T_M$ such that $1 < d_{T_M}(v) < \Delta$ and $1 < d_{T_M}(u) < \Delta$. Without loss of generality, assume $x_v \geqslant x_u$. Let $a$ be a neighbor of $u$ such that $a$ is further from $v$ than $u$. Let $T'$ be the weighted tree obtained from $T_M$ by deleting the edge $ua$ and adding the new edge $va$ such that

$$w_{T'}(va) = w_{T_M}(ua), \quad w_{T'}(e) = w_{T_M}(e), \quad e \in E(T_M) - \{ua\}.$$

Then $T' \in \Omega(\Delta, W_n)$, and by Lemma 2.2 we have $\rho(T') > \rho(T_M)$, a contradiction with the assumption of $T_M$. Thus $T_M$ has at most a vertex $u$ such that $1 < d_{T_M}(u) = t < \Delta$. Let $s, p, q$ denote the numbers of vertices with degrees $\Delta$, $t$ and 1, respectively. Then $p = 0, 1$ and $s, p, q, t$ satisfy $s + p + q = n$, $s\Delta + pt + q = 2(n - 1)$. These indicate that the degree sequence of $T_M$ is $D_n = (\Delta^{(s)}, t^{(p)}, 1^{(q)})$. It follows that $T_M \in \Gamma(D_n, W_n)$. So by Theorem 2.7 the WBFD-tree in $\Gamma(D_n, W_n)$ is the unique graph in $\Omega(\Delta, W_n)$ having the largest adjacent spectral radius. □

Let $P(s, W_n)$ be the set of all weighted trees with a given positive weight set $W_n$ and $s$ pendant vertices.

**Theorem 2.9.** *The WBFD-tree in* $\Gamma((s^{(1)}, 2^{(n-s-1)}, 1^{(s)}), W_n)$ *is the unique graph in* $P(s, W_n)$ *having the largest adjacent spectral radius.*

**Proof.** Let $T_M$ be a weighted tree in $P(s, W_n)$ with the largest adjacent spectral radius.

If $s = 2$ then $T_M$ is a path. So the degree sequence of $T_M$ is $D_n = (2^{(n-2)}, 1^{(2)})$.

Let $s \geqslant 3$. Suppose that there are two vertices $v$ and $u$ of $T_M$ with $3 \leqslant d_{T_M}(v) \leqslant s$ and $3 \leqslant d_{T_M}(u) \leqslant s$. Without loss of generality, assume $x_v \geqslant x_u$. Let $a$ be a neighbor of $u$ such that $a$ is further from $v$ than $u$. Let $T'$ be the weighted tree obtained from $T_M$ by deleting the edge $ua$ and adding the new edge $va$ such that

$$w_{T'}(va) = w_{T_M}(ua), \quad w_{T'}(e) = w_{T_M}(e), \quad e \in E(T_M) - \{ua\}.$$

Then $T' \in P(s, W_n)$, and by Lemma 2.2 we have $\rho(T') > \rho(T_M)$, a contradiction with the assumption of $T_M$. Thus $T_M$ has a unique vertex $v$ with $d_{T_M}(v) \geqslant 3$. So $T_M$ is a weighted starlike tree with the center $v$. It follows that $d_{T_M}(v) = s$ and the distinct degrees of $T_M$ are 1, 2 and $s$. It is easy to see that the number of vertices with the degree 2 is $n - s - 1$. So the degree sequence of $T_M$ is $D_n = (s^{(1)}, 2^{(n-s-1)}, 1^{(s)})$.

The discussions above indicate that $T_M \in \Gamma(D_n, W_n)$. By Theorem 2.7 the WBFD-tree in $\Gamma(D_n, W_n)$ is the unique graph in $P(s, W_n)$ having the largest adjacent spectral radius. $\square$

A vertex adjacent to a pendant vertex is called a *quasi-pendant vertex*. Let $Q(s, W_n)$ be the set of all weighted trees with a given positive weight set $W_n$ and $s$ quasi-pendant vertices. Let $T_{n,s}$ be the WBFD-tree in $\Gamma((s^{(1)}, 2^{(n-s-1)}, 1^{(s)}), W_n)$. By Theorem 2.9 $T_{n,s}$ is the unique weighted tree in $P(s, W_n)$ having the largest adjacent spectral radius.

**Theorem 2.10.** $T_{n,n-s}$ *is the unique graph in* $Q(s, W_n)$ *having the largest adjacent spectral radius.*

**Proof.** Let $T_M$ be a weighted tree in $Q(s, W_n)$ with the largest adjacent spectral radius. Suppose that $T_M$ has a vertex $v$ that is neither a pendant vertex nor a quasi-pendant vertex. Let $z_1$ be any pendant vertex of $T_M$ and let the path from $z_1$ to $v$ be

$$P = z_1 z_2 z_3 \cdots z_r z_{r+1} \cdots z_s,$$

where $z_s = v,\ z_i z_{i+1} \in E(T_M),\ i = 1, 2, \ldots, s-1$. By the assumptions of $z_1$ and $z_s$ there must be a $r$ such that $z_r$ is a quasi-pendant vertex and $z_{r+1}$ is neither a pendant vertex nor a quasi-pendant vertex. Let $T'$ be the weighted tree obtained from $T_M$ by deleting the edge $z_r z_{r+1}$, identifying $z_r$ and $z_{r+1}$ (denote the new vertex by $z$), and adding the new edge $zz'$ such that

$$w_{T'}(zz') = w_{T_M}(z_r z_{r+1}), \quad w_{T'}(e) = w_{T_M}(e), \quad e \in E(T_M) - \{z_r z_{r+1}\}.$$

Then $T' \in Q(s, W_n)$, and from Lemma 2.2 we have $\rho(T') > \rho(T_M)$, a contradiction with the assumption of $T_M$. Therefore, each vertex of $T_M$ is a pendant vertex or a quasi-pendant vertex. It follows that $T_M$ has $n - s$ pendant vertices, i.e., $T_M \in P(n - s, W_n)$. By Theorem 2.9 $T_{n,n-s}$ is the unique graph in $Q(s, W_n)$ having the largest adjacent spectral radius. $\square$

## 3. On the weighted trees with the largest Laplacian spectral radius

For any weighted graph $G$, we call the matrix $R(G) = W(G) + A(G)$ the *signless Laplacian matrix* of $G$. The largest eigenvalue of $R(G)$ is called the *signless Laplacian spectral radius* of $G$, denoted by $\nu(G)$. Since $R(G)$ is nonnegative, there is a nonnegative eigenvector corresponding to $\nu(G)$. In particular, when $G$ is connected, $R(G)$ is irreducible and by the well-known Perron–Frobenius Theorem, $\nu(G)$ is simple and there is a unique positive unit eigenvector. We shall refer to such an eigenvector $x$ as the *signless Laplacian Perron vector* of $G$ and still denote the component corresponding to the vertex $v$ of $G$ by $x_v$.

The following are three results about the perturbations of signless Laplacian spectral radius of weighted graphs [19,20].

**Lemma 3.1.** *Let $uv$ and $ab$ be distinct edges of a weighted connected graph $G$ and let $x$ be the signless Laplacian Perron vector of $G$. For $0 < \delta \leqslant w_G(uv)$, let $G^1$ be the weighted graph obtained from $G$ such that*

$$w_{G^1}(uv) = w_G(uv) - \delta, \quad w_{G^1}(ab) = w_G(ab) + \delta,$$
$$w_{G^1}(e) = w_G(e), \quad e \in E(G) - \{ab, uv\}.$$

*If $x_u + x_v \leqslant x_a + x_b$, then $\nu(G) < \nu(G^1)$.*

**Lemma 3.2.** *Let $u$ and $v$ be two distinct vertices of a weighted connected graph $G$. Let $u_1, u_2, \ldots, u_s (u_i \neq v, s \neq 0)$ be some vertices of $N_G(u) \backslash N_G(v)$ and let $x$ be the signless Laplacian Perron vector of $G$. Let $G^2$ be the weighted graph obtained from $G$ by deleting edges $uu_j$ and adding edges $vu_j$ such that*

$$w_{G^2}(vu_j) = w_G(uu_j), \quad w_{G^2}(e) = w_G(e), \quad e \neq uu_j, \quad j = 1, 2, \ldots, s.$$

*If $x_u \leqslant x_v$, then $\nu(G) < \nu(G^2)$.*

**Lemma 3.3.** *Let $ab$, $uv$, $vb$ and $au$ be distinct edges of a weighted connected graph $G$ and let $x$ be the signless Laplacian Perron vector of $G$. For $0 < \theta \leqslant w_G(ab)$, $0 < \delta \leqslant w_G(uv)$, let $G^3$ be the weighted graph obtained from $G$ such that*

$$w_{G^3}(ab) = w_G(ab) - \theta, \quad w_{G^3}(au) = w_G(au) + \theta, \quad w_{G^3}(uv) = w_G(uv) - \delta,$$
$$w_{G^3}(vb) = w_G(vb) + \delta, \quad w_{G^3}(e) = w_G(e), \quad e \in E(G) \backslash \{ab, uv, vb, au\}.$$

*If $(x_u - x_b)[(2x_a + x_b + x_u)\theta - (2x_v + x_b + x_u)\delta] \geqslant 0$, then $\nu(G) \leqslant \nu(G^3)$.*

*In addition, $\nu(G) = \nu(G^3)$ if and only if $x_u = x_b$ and $(2x_a + x_b + x_u)\theta = (2x_v + x_b + x_u)\delta$.*

According to Lemmas 3.1–3.3, in the similar way to obtain Theorem 2.7, we can get the following result.

**Theorem 3.4.** *The WBFD-tree is the unique graph in $\Gamma(D_n, W_n)$ having the largest signless Laplacian spectral radius.*

For a weighted bipartite graph $G$, since $L(G)$ and $R(G)$ have the same spectrum [20], by Theorem 3.4 we immediately obtain another main result in this paper.

**Theorem 3.5.** *The WBFD-tree is the unique graph in $\Gamma(D_n, W_n)$ having the largest Laplacian spectral radius.*

In the similar way to obtain Theorems 2.8–2.10, we can obtain the following three results, where Theorem 3.7 has appeared in [20].

**Theorem 3.6**

(1) *If $\Delta = 2$ then the WBFD-tree in $\Gamma((2^{(n-2)}, 1^{(2)}), W_n)$ is the unique graph in $\Gamma(\Delta, W_n)$ having the largest Laplacian spectral radius.*
(2) *If $\Delta \geqslant 3$ then the WBFD-tree in $\Gamma((\Delta^{(s)}, t^{(p)}, 1^{(q)}), W_n)$ is the unique graph in $\Gamma(\Delta, W_n)$ having the largest Laplacian spectral radius, where $1 < t < \Delta$, $0 \leqslant p \leqslant 1$ and $s$, $p$, $q$, $t$ satisfy $s + p + q = n$, $s\Delta + pt + q = 2(n-1)$.*

**Theorem 3.7.** *The WBFD-tree in $\Gamma((s^{(1)}, 2^{(n-s-1)}, 1^{(s)}), W_n)$ is the unique graph in $P(s, W_n)$ having the largest Laplacian spectral radius.*

**Theorem 3.8.** *$T_{n,n-s}$ is the unique graph in $Q(s, W_n)$ having the largest Laplacian spectral radius.*

## Acknowledgements

The author would like to thank the anonymous referees very much for giving many valuable suggestions, corrections and comments towards improving this paper.

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